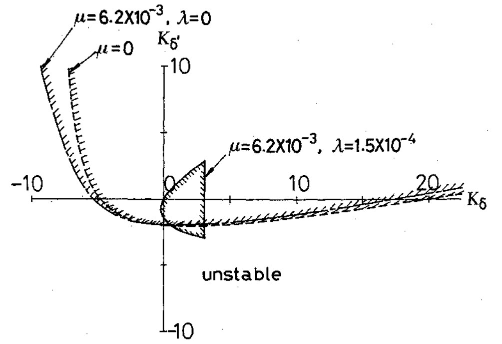

Fig. 2 Stable regions of controlled system in the K_δ–K_δ' plane ($R = 6.6 \times 10^6$ m, $l_s = 10^5$ m, $\omega = 1.18 \times 10^{-3}\,s^{-1}$, $\Omega = 7 \times 10^{-5}\,s^{-1}$, $K_\gamma = 7.359$, $K_\gamma' = 5.368$).

In the case of no atmosphere, i.e., $\mu = 0$, the optimal control gains that minimize

$$E(\{q\}^T[Q]\{q\} + R_u u^2) \tag{26}$$

for the linearized system is derived as case A of Table 1 based on the same weighting matrix as those used in Ref. 4, where $E(\ )$ denotes the expectation operator. The values obtained for the gains coincide with those of Ref. 4. When the values of two control gains K_γ and K_γ' are fixed to these values, the stable region in the $K_\delta - K_\delta'$ plane is obtained from Eqs. (17-25) and the Routh-Hurwitz's stability criterion as the upper region than the broken line of Fig. 2, which also coincides with the result of Ref. 4. For the case of $\mu = 6.2 \times 10^{-3}$, $\lambda = 0$, i.e., fictious case of locally uniform atmosphere, the stable region is obtained as the upper region than the one-dot chain line of Fig. 2. These two stable regions are similar to each other. However, in the case of $\mu = 6.2 \times 10^{-3}$ and $\lambda = 1.5 \times 10^{-4}$ m^{-1}, i.e., the realistic atmosphere with density gradient, the stable region is limited to the inside of the very small domain shown in Fig. 2 by the solid line. These facts indicate the significance of the effects of the atmospheric density gradient on the stability of controlled subsatellites.

The closed-loop eigenvalues are also listed in Table 1. They indicate that the system is stable in case A. Transient motions of the system obtained by a numerical integration of Eqs. (1-7) show well-controlled transient responses, although they are not shown here because of the need for brevity.

In case B of Table 1, the same gains as case A are adopted to the system with fictitious atmosphere whose density is locally uniform. The eigenvalues are almost the same as those of case A, and the system is stable. This fact indicates that the effects of the atmosphere are small if its density were uniform.

Case C is the case where the same gains as case A are adopted to the subsatellite deployed in the real atmosphere with density gradient. Unlike cases A and B, one of the real parts of the eigenvalues of case C is positive, indicating that the system is unstable. Reference 3 lists the eigenvalues of an uncontrolled subsatellite obtained, based on the same parameter values as case C, and a realistic longitudinal stiffness of the tether. It should be noted that the positive real part of an eigenvalue of case C, 1.153, is much larger than that of uncontrolled subsatellite, 3.517×10^{-2}, which can be obtained by normalizing the result of Ref. 3. This fact indicates that the effects of the density gradient can be far more significant for the controlled system than for the uncontrolled one, and a control system designed without any account of the atmospheric density gradient can make the system quite unstable. In Ref. 3, it has been suggested that lower longitudinal stiffness of the tether unstabilizes the system. Since the gain K_γ corresponds to the longitudinal stiffness, the previous relatively strong instability seems to be caused by the relatively small control gain K_γ.

The optimal linear quadratic Gaussian (LQG) control gains can be obtained based on the actual values of μ and λ, i.e., actual atmospheric parameters, as case D in Table 1. In this case, the eigenvalues indicate that the system is stable. As can be expected from the preceding investigation, the longitudinal stiffness K_γ is very large in this stable system. Transient responses, which are not shown in this Note because of lack of space, indicate that the system is relatively well controlled, although a relatively large control force suggests the difficulty of control in the atmospheric density gradient.

Case E and case F are the examples where the Rupp's control laws[5] are adopted. The former is the case of $\xi = 1$, and the latter is the case of $\xi = 2$ according to the notation of Ref. 5. The eigenvalues indicate that the motion is unstable in case E and stable in case F. The transient responses of case F, which are not shown here, indicate that the system is not as well controlled as is in case D although the closed-loop system of case F is stable.

IV. Concluding Remarks

The effects of the atmospheric density gradient on the in-plane motion of tension-controlled tethered subsatellites deployed in a low-altitude orbit are investigated, based on a simple model. It is shown that the effect on the stability of the controlled system can be more significant than on that of the uncontrolled system. A control system designed without any account of the atmospheric density gradient can greatly unstabilize the closed-loop system. Therefore, it is indispensible to consider the effects of the density gradient in the design of the control system of tethered subsatellites deployed in low-altitude orbits.

Although the present Note has clarified the effects of the atmospheric density gradient qualitatively based on the simplified model, further investigations with a more exact model are still required.

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Optimal Terminal Maneuver for a Cooperative Impulsive Rendezvous

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I. Introduction

IN this Note, the optimal terminal maneuver is determined for a cooperative impulsive rendezvous of two space vehi-

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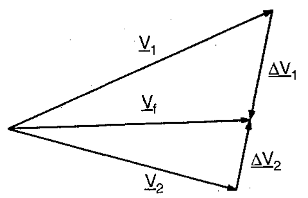

Fig. 1. Velocity changes for cooperative rendezvous.

cles. The term "cooperative rendezvous" implies that each vehicle is active, i.e., capable of providing part or all of the total velocity change required for the rendezvous. This maneuver, which occurs at the point of interception of the two vehicles, must be performed in an optimal manner to be part of an overall optimal cooperative rendezvous solution. The common velocity vector of the two vehicles after the rendezvous is unspecified, and is therefore free to be optimized.

Optimal rendezvous implies that the sum of the final masses of the two vehicles is maximized. This is synonymous with minimum total propellant consumption, but it is not the same as minimum total ΔV because the masses and exhaust velocities of the two vehicles are in general different.

The propellant mass fraction of each vehicle is assumed to be constrained between a minimum value of zero and a specified maximum value that is less than unity. In the case of only one active vehicle, this is a moot point. The active vehicle either has enough propellant to perform the rendezvous or it does not. By contrast, the two-active-vehicle case is more interesting. If each vehicle has enough propellant to perform the rendezvous, the more propellant-efficient vehicle will provide the total required velocity change. However, if the more efficient vehicle does not have enough propellant to provide the total ΔV, the other vehicle must provide some, or in some cases, all of the velocity change. Illustrations of these cases are provided by the numerical examples at the end of this Note.

II. Analysis

The problem of determining how much of the total required velocity change should be provided by each vehicle is a straightforward constrained optimization problem as will be shown. However, the solution is interesting because of its complexity. There exists either two or three distinct solutions to the necessary conditions for the constrained minimum propellant solution. Of these, either one or two satisfy the sufficient conditions for a constrained local minimum; those that do not represent constrained local maximum solutions and obviously should be avoided. When two constrained local minima exist, in some cases the global minimum is immediately known; in other cases a direct cost comparison of the two local minima must be made.

The optimization problem can be formulated in terms of the important parameters of the two vehicles at intercept, namely, the masses m_1 and m_2, the velocities V_1 and V_2, and the effective exhaust velocities c_1 and c_2. The total propellant consumption to be minimized is $L = \Delta m_1 + \Delta m_2$.

Because the final velocity after the rendezvous V_f is unspecified, it is evident that its optimal value must be a convex combination of the vehicle velocities at interception V_1 and V_2 as shown in Fig. 1 with the final velocity given by

$$V_f = \sigma V_1 + (1-\sigma)V_2, \qquad 0 \le \sigma \le 1 \tag{1}$$

To see this, denote $|V_2 - V_1|$ as K, and note that the total required velocity change is then $\Delta V_1 + \Delta V_2 = K$. Any final velocity other than a convex combination as shown in Fig. 1 will require $\Delta V_1 + \Delta V_2 > K$ and therefore will require a larger propellant combustion.

The well-known relationship between propellant consumption Δm_j for each vehicle and the corresponding velocity change ΔV_j is given by

$$\frac{\Delta m_j}{m_j} = 1 - e^{-\Delta V_j / c_j} \tag{2}$$

Using Eq. (2) and the constraint $\Delta V_1 + \Delta V_2 = K$, the function $L = \Delta m_1 + \Delta m_2$ to be minimized can be expressed in terms of a single variable. If one denotes ΔV_1 as x, ΔV_2 is then $K - x$, and the function to be minimized is

$$L(x) = m_1(1 - e^{-x/c_1}) + m_2(1 - e^{-(K-x)/c_2}) \tag{3}$$

The propellant mass fraction constraints can be expressed as

$$0 \le \frac{\Delta m_j}{m_j} \le \beta_j < 1 \tag{4}$$

which can also be described using Eq. (2) in terms of the corresponding maximum allowed velocity changes a_j:

$$a_j = -c_j \, \ell n(1 - \beta_j) \tag{5}$$

Equations (2), (4), and (5) provide the complete set of constraints on x as

$$p_1 \le x \le p_2 \tag{6}$$

where

$$p_1 = \max(0, K - a_2) \tag{7a}$$

$$p_2 = \min(K, a_1) \tag{7b}$$

From Eq. (6), it is evident that a rendezvous solution exists if and only if $p_1 \le p_2$, i.e.,

$$a_1 + a_2 \ge K$$

The function $L(x)$ in Eq. (3) to be minimized can now be augmented by the constraints of Eq. (6) to form

$$H(x, \lambda_1, \lambda_2) = L(x) + \lambda_1(p_1 - x) + \lambda_2(x - p_2) \tag{8}$$

The necessary conditions for a constrained minimum are given by Ref 1:

$$\frac{\partial H}{\partial x} = L'(x) - \lambda_1 + \lambda_2 = 0$$

with

$$\lambda_j \ge 0, \qquad j = 1,2 \tag{9}$$

and

$$\lambda_1 = 0 \qquad \text{if } p_1 < x$$

$$\lambda_2 = 0 \qquad \text{if } x < p_2$$

where

$$L'(x) = \frac{m_1}{c_1} e^{-x/c_1} - \frac{m_2}{c_2} e^{-(K-x)/c_2} \tag{10}$$

Determining all the solutions to the necessary conditions given by Eq. (9) is greatly expedited by observing that the function $L(x)$ is everywhere concave:

$$L''(x) = -\frac{m_1}{c_1^2} e^{-x/c_1} - \frac{m_1}{c_1^2} e^{-(K-x)/c_2} < 0$$

The constrained minimum solution x^* can then be determined immediately for two cases:

Case I:

$$\text{If } L'(p_1) \le 0 \qquad \text{then } x^* = p_1 \tag{11}$$

Case II:

$$\text{If } L'(p_2) \ge 0 \qquad \text{then } x^* = p_1 \tag{12}$$

Because $L'' < 0$, case I corresponds to a constrained global maximum of L at $x = p_1$ and the desired constrained global minimum at $x = p_2$. The necessary conditions of Eq. (9) are satisfied by $\lambda_1 = 0$ and $\lambda_2 = -L'(p_1) \ge 0$. Case II represents a constrained global maximum at $x = p_2$ and the desired constrained global minimum at $x = p_1$. The necessary conditions are satisfied by $\lambda_2 = 0$ and $\lambda_1 = L'(p_2) \ge 0$.

The condition of Eq. (11) for case I can be stated explicitly using the expression for L' in Eq. (10) as

$$K \le C_2 \ell n \alpha + \left(1 + \frac{c_2}{c_1}\right) p_1 \tag{13}$$

where $\alpha = (m_2/m_1)/(c_2/c_1)$. Note that when $p_1 = 0$, i.e., K is sufficiently small that the propellant mass fraction constraint of vehicle 1 is inactive, Eq. (13) can be satisfied only if $\alpha > 1$. The condition of Eq. (12) for case II requires that

$$K \ge c_2 \ell n \alpha + \left(1 + \frac{c_2}{c_1}\right) p_2 \tag{14}$$

Note that when $p_2 = K$, i.e., K is sufficiently small that the propellant mass fraction constraint of vehicle 2 is inactive, Eq. (14) can be satisfied only if $\alpha < 1$.

Besides cases I and II, the only other possible situation is Case III, for which $L'(p_1) > 0$ and $L'(p_2) < 0$. In this case one solution to the necessary conditions of Eq. (9) is $\lambda_1 = \lambda_2 = 0$ (both constraints inactive) and $L'(x^*) = 0$. However, because $L'' < 0$, this stationary value is a global maximum solution which must be discarded. The desired solution is one of the two constrained minima which exist at $x = p_1$ and p_2, for which the respective solutions to the necessary conditions are $\lambda_2 = 0$ along with $\lambda_1 = L'(p_1) \ge 0$ and $\lambda_1 = 0$ along with $\lambda_2 = -L'(p_2) \ge 0$. Whichever local minimum provides the smaller value of L is the global constrained minimum.

III. Numerical Examples

The four numerical examples that follow illustrate the variety and complexity of the optimal solutions. In all of the examples, $m_2/m_1 = 1.5$ and $c_2/c_1 = 1.25$, resulting in $\alpha = 1.2$. In the first example, $K/c_1 = 0.2$ with the propellant mass fractions given by $\beta_1 = 0.3$ and $\beta_2 = 0.2$. Because the required velocity change K is sufficiently small, $p_1 = 0$ and $p_2 = K$ in Eq. (7) and the condition of Eq. (11) representing case I is satisfied. Thus, $x^* = K$ is the constrained optimal solution, indicating that $\Delta V_1 = K$ and $\Delta V_2 = 0$. All of the velocity change is made by vehicle 1, which has both the smaller mass and a smaller exhaust velocity.

As a second example, consider the large velocity change $K/c_1 = 1$ along with the mass fraction constraints $\beta_1 = 0.65$ and $\beta_2 = 0.52$. In this case, the mass fraction constraint of vehicle 2 is active and $p_1 = K - a_2 = 0.0825c_1$ in Eq. (7). The value of p_2 is K, because the mass fraction constraint of vehicle 1 is inactive. Because $L'(p_1) > 0$ and $L'(p_2) < 0$, this represents case III, and a direct comparison of the cost yields $x^* = p_2 = K$, indicating $\Delta V_1 = K$ and $\Delta V_2 = 0$.

In the third example, $K/c_1 = 0.8$ and $\beta_1 = \beta_2 = 0.5$. In this case, $p_1 = 0$ and $p_2 = a_1 = 0.693c_1$ because only the vehicle 2 mass fraction constraint is active. This is an example of case III for which the optimal solution is $x^* = 0.693c_1$, indicating that both vehicles provide velocity change: $\Delta V_1 = 0.693c_1$, and $\Delta V_2 = 0.107c_1$.

The fourth and final example is the same as the second example except that $\beta_1 = 0.3$ rather than 0.65. This has the effect of making both propellant mass fraction constraints active. This effect automatically requires both vehicles to provide velocity change because neither has enough propellant to perform the total maneuver. When the costs of the two constrained local minima are compared for this case III solution, the optimal solution is $x^* = p_1 = 0.0825c_1$. Thus, $\Delta V_1 = 0.0825c_1$ and $\Delta V_2 = 0.9175c_1$, which is a surprising result compared to the second example. In that example, vehicle 2 did not have enough propellant to perform the rendezvous and vehicle 1 provided the total velocity change. When neither vehicle has enough propellant to perform the rendezvous, the optimal solution requires that vehicle 2 provides over 90% of the velocity change.

The explanation of this apparent paradox lies in the fact that in the fourth example, vehicle 2 was forced to provide part of the velocity change because vehicle 1 did not have enough propellant. The mass decrease of vehicle 2 incurred in making up the velocity change deficit of vehicle 1 is enough to make vehicle 2 the more propellant-efficient vehicle of the two. It is therefore optimal to have vehicle 2 make up the entire deficit velocity change of vehicle 1.

IV. Concluding Remarks

The apparently simple problem of optimal cooperative rendezvous of two active space vehicles yields solutions which are interesting due to their complexity. If neither propellant mass fraction constraint is active, one vehicle provides all of the required velocity change. If both mass fraction constraints are active, there are two possibilities: both vehicles provide velocity change, or, if the required velocity change is too large, no solution exists. In the case of one active propellant mass constraint, the optimal solution can require either one or both vehicles to provide velocity change.

Acknowledgment

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Sequential Design of Discrete Linear Quadratic Regulators via Optimal Root-Locus Techniques

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I. Introduction

THE asymptotic behavior of the optimal root-loci (frequency-domain approach) of linear time-invariant continuous-time control systems has been discussed by Chang[1] and Kalman[2] for the single-input case, and the multi-

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